

# EXPONENTIAL DEPHASING OF OSCILLATORS IN THE KINETIC KURAMOTO MODEL

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**ABSTRACT.** We study the kinetic Kuramoto model for coupled oscillators with coupling constant below the synchronization threshold. We manage to prove that, for any analytic initial datum, if the interaction is small enough, the order parameter of the model vanishes exponentially fast, and the solution is asymptotically described by a free flow. This behavior is similar to the phenomenon of Landau damping in plasma physics. In the proof we use a combination of techniques from Landau damping and from abstract Cauchy-Kowalewska theorem.

## 1. INTRODUCTION

The Kuramoto model is a mean-field model of coupled oscillators proposed by Kuramoto to describe synchronization phenomena (see [14],[18],[13]). Any oscillator has a phase  $\vartheta$ , that can be considered defined mod  $2\pi$ , i.e. in the one-dimensional torus  $\mathcal{T}$ , and a “natural frequency”  $\omega \in \mathbb{R}$ . In the kinetic limit the model reads as

$$(1.1) \quad \begin{cases} \partial_t f(t, \vartheta, \omega) + \partial_{\vartheta}(v(t, \vartheta, \omega) f(t, \vartheta, \omega)) = 0 \\ v(t, \vartheta, \omega) = \omega - \mu \int_{\mathcal{T} \times \mathbb{R}} \sin(\vartheta - \vartheta') f(t, \vartheta', \omega') d\vartheta' d\omega' \end{cases},$$

where  $f(t, \vartheta, \omega)$  is a probability density in  $\mathcal{T} \times \mathbb{R}$ ,  $\int_{\mathcal{T} \times \mathbb{R}} \sin(\vartheta - \vartheta') f(t, \vartheta', \omega') d\vartheta' d\omega'$  is the mean field interaction term, and  $\mu > 0$  is the coupling constant. The distribution of the natural frequencies is  $g(\omega) = \int_{\mathcal{T}} f(t, \vartheta, \omega) d\vartheta$ , which is a conserved quantity.

It can be useful to represent the system (1.1) in the unitary circle of the complex plane by considering the oscillators as particles with position  $e^{i\vartheta}$ . The center of mass is in the point

$$(1.2) \quad R(t)e^{i\varphi(t)} = \int f(t, \vartheta, \omega) e^{i\vartheta} d\vartheta d\omega.$$

$R$  and  $\varphi$  are the “order parameters” of the model. By this notation the coupling term can be rewritten as

$$(1.3) \quad \int_{\mathcal{T} \times \mathbb{R}} \sin(\vartheta - \vartheta') f(t, \vartheta', \omega') d\vartheta' d\omega' = R(t) \sin(\vartheta - \varphi(t));$$

so that the mean field interaction between the particles can be read as an attraction towards the phase of the center of mass, modulated by  $R(t)$ .

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Existence and uniqueness results for the system (1.1) are obtained in [15], where the (1.1) is rigorously derived by doing the kinetic limit of the particle model introduced by Kuramoto.

The model has been intensively studied in the case where  $g$  has compact support and the coupling constant  $\mu$  is sufficiently large to observe the asymptotic synchronization phenomenon (see [7],[6],[11] and [2],[8] for the simpler case of  $g(\omega)$  being a Dirac delta). The asymptotic behavior of the solutions can be more complex: in particular, the total synchronization is impossible if  $g$  has not compact support, and in this case it is expected a partial synchronization (see [18]).

On the other hand, a completely different asymptotic behavior is suggested in [19], in which, for small value of  $\mu$ , it is shown a Landau-damping type results for a linearized model: the order parameter of a perturbation of the constant phase density relaxes to zero as the time goes to infinity, and the solution becomes an “incoherent state”. This behavior has been proved in the recent work [10] for the non linear model, where a full Landau-damping type result is obtained in the case of Sobolev regularity, showing that  $R$  vanishes polynomially in time.

In this work we show, for small values of  $\mu$ , the asymptotic dephasing of the solutions of the nonlinear equation (1.1) in the case of analytic initial data; in particular, we prove that  $R(t)$  vanishes exponentially fast. The techniques we use take inspiration from the works on Landau damping for the Vlasov equation (see [12],[17]) and on the damping for the Euler equations (see [1]), and mostly from the more recent work [9], in which it is shown the Landau damping in Sobolev norm for the Vlasov HMF model. Unlike the case of Sobolev regularity, the estimates in analytic norms do not close, so we adapt to this case the abstract Cauchy-Kowalewskaya Theorem techniques, in a way that allow us to obtain globally in time existence, while the abstract Cauchy-Kowalewskaya Theorem gives only finite time existence.

We remark that the phenomenon of dephasing of the solutions of the Kuramoto model has been studied in the recent work [10], in which it is shown a Landau-damping type result for eq. (1.1) in Sobolev norms, and in the work [3] in which it is shown the existence of dephasing solutions with a prescribed asymptotic behavior, with analytical or Sobolev regularity.

## 2. DEPHASING

In terms of the order parameters, the kinetic Kuramoto equation reads as

$$(2.1) \quad \begin{cases} \partial_t f(t, \vartheta, \omega) + \partial_\vartheta(v(t, \vartheta, \omega)f(t, \vartheta, \omega)) = 0 \\ v(t, \vartheta, \omega) = \omega - \mu R(t) \sin(\vartheta - \varphi(t)) \\ R(t)e^{i\varphi(t)} = \int_{\mathcal{T} \times \mathbb{R}} f(t, \vartheta, \omega) e^{i\vartheta} d\vartheta d\omega. \end{cases}.$$

If  $\mu = 0$ , the solutions of eq. (2.1) is

$$f(t, \vartheta, \omega) = f_0(\vartheta - \omega t, \omega)$$

where  $f_0$  is the initial datum. Our aim is to show that, if  $f_0(\vartheta, \omega)$  is bounded in some analytical norm, and  $\mu > 0$  is sufficiently small, there exists an asymptotic state  $h_\infty$  such that

$$f(t, \vartheta + \omega t, \omega) \rightarrow h_\infty(\vartheta, \omega)$$

exponentially fast. In other words, the solutions  $f(t, \vartheta, \omega)$  asymptotically approaches the incoherent state  $h_\infty(\vartheta - \omega t, \omega)$ , i.e. a function transported by the free flow. In

this sense, the solutions show an asymptotic dephasing. The key ingredient of the proof is the exponential decay of the order parameter  $R(t)$ , which, as noted in [16], can only be obtained for analytic initial data.

To state precisely our result, we define  $h(t, \vartheta, \omega) = f(t, \vartheta + \omega t, \omega)$ , with initial datum  $h(0, \vartheta, \omega) = f_0(\vartheta, \omega)$  and which verifies, from eq. (2.1), the equation

$$(2.2) \quad \partial_t h(t, \vartheta, \omega) = -\mu R(t) \partial_{\vartheta} (\sin(\vartheta + \omega t - \varphi(t)) h(t, \vartheta, \omega)).$$

Defining for  $k \in \mathbb{Z}$  and  $\eta \in \mathbb{R}$

$$\hat{h}_k(t, \eta) = \frac{1}{2\pi} \int_{\mathcal{T} \times \mathbb{R}} d\vartheta d\omega h(t, \vartheta, \omega) e^{-i\vartheta k - i\omega \eta}$$

eq. (2.2) in Fourier space is

$$(2.3) \quad \partial_t \hat{h}_k(t, \eta) = \mu \widehat{\mathbf{L}_t h}(k, \eta), \quad \text{where}$$

$$(2.4) \quad \widehat{\mathbf{L}_t f}(k, \eta) \doteq k \sum_{m=\pm 1} \frac{m}{2} z_m(t) \hat{g}_{k-m}(\eta - mt),$$

where the order parameters read as

$$z_{\pm 1}(t) = \hat{h}_{\pm 1}(t, \pm t) = R(t) e^{\mp i\varphi(t)}, \quad |z(t)| = R(t).$$

Integrating in time eq.s (2.3) we have

$$(2.5) \quad \hat{h}_k(t, \eta) = \hat{h}_k(0, \eta) + \mu k \sum_{m \in \pm 1} \frac{m}{2} \int_0^t z_m(s) \hat{h}_{k-m}(s, \eta - ms) ds.$$

In the sequel, we consider separately the evolution in time of the order parameter  $z_{\pm 1}$ , and of the function  $h$ ; in this sense  $\mathbf{L}_t$  can be considered a linear operator in  $h$ .

We use analytical norms for  $h$  so, to make the notation lighter, we give the following definitions:

$$(2.6) \quad \langle t \rangle = (1 + t^2)^{\frac{1}{2}}, \quad \langle k, \eta \rangle = (1 + k^2 + \eta^2)^{\frac{1}{2}};$$

note that  $\langle \cdot \rangle$  verifies the triangular inequality, as follows from easy calculation:

$$(2.7) \quad \langle k_1 + k_2, \eta_1 + \eta_2 \rangle \leq \langle k_1, \eta_1 \rangle + \langle k_2, \eta_2 \rangle.$$

For  $\lambda, p \geq 0$ , we define the weight

$$(2.8) \quad A_k^{\lambda, p}(\eta) = e^{\lambda \langle k, \eta \rangle} \langle k, \eta \rangle^p,$$

and the norms:

$$(2.9) \quad \|f\|_{\lambda, p} = \sup_{k \in \mathbb{Z}, \eta \in \mathbb{R}} A_k^{\lambda, p}(\eta) |\hat{f}_k(\eta)|.$$

We call  $\mathcal{X}_{\lambda, p}$  the space of function  $f$  with finite  $\|f\|_{\lambda, p}$  norm.

Using this norm, it is easy to show that if  $\mu = 0$ , we obtain the exponential decay of the order parameter  $z_{\pm 1}(t)$ . Let us first obtain an equation for  $z_1 = R(t) e^{-i\varphi(t)}$  setting  $k = 1$  and  $\eta = t$  in (2.5)

$$(2.10) \quad z_1(t) = \hat{h}_1(0, t) + \mu \sum_{m=\pm 1} \frac{m}{2} \int_0^t z_m(s) \hat{h}_{1-m}(s, t - ms) ds,$$

where  $h(0, \vartheta, \eta)$  is the initial datum  $f_0(\vartheta, \eta)$ . Choosing  $\lambda, p \geq 0$ , the first term, due to the free flow, is bounded by

$$\|f_0\|_{\lambda, p} e^{-\lambda \langle 0, t \rangle} \langle 0, t \rangle^p \leq C e^{-\lambda t} \langle t \rangle^p \|f_0\|_{\lambda, p}$$

where, here and in the following,  $C$  is a suitable time independent constant. This estimate suggests that we can control the quantities

$$(2.11) \quad r_{\lambda,p}(t) = |z_{\pm 1}(t)|e^{\lambda t}\langle t \rangle^p.$$

An uniform in time bound of  $r_{\lambda,p}(t)$  is equivalent to an exponential decay of  $R(t)$ . The aim of this paper is to show that, if  $\mu$  is sufficiently small, the other terms in (2.10) do not prevent a uniform estimate for  $r_{\lambda,p}$ .

### 3. A-PRIORI ESTIMATES

In the case  $\mu > 0$ ,  $r_{\lambda,p}(t)$  can be estimated as in the following proposition.

**Proposition 3.1.** *For  $\lambda, p \geq 0$*

$$(3.1) \quad \begin{aligned} r_{\lambda,p}(t) \leq & C\|f_0\|_{\lambda,p} \\ & + \mu C\|f_0\|_{\lambda,p} \int_0^t r_{\lambda,p}(s) \left( \frac{1}{\langle s \rangle^p} + \frac{1}{\langle t-s \rangle^p} \right) ds + \mu C \int_0^t r_{\lambda,p}(s) \frac{\|h(s)\|_{\lambda,p}}{\langle s \rangle^p} ds. \end{aligned}$$

*Proof.* The first term of the r.h.s. of (2.10) is simply bounded by

$$\|f_0\|_{\lambda,p} e^{\lambda\langle 0,t \rangle} \langle 0,t \rangle^p \leq C e^{\lambda t} \langle t \rangle^p \|f_0\|_{\lambda,p}.$$

Since  $\hat{f}_0(t, \eta) = \hat{f}_0(0, \eta)$ , the term with  $m = 1$  is

$$\int_0^t z_1(s) \hat{f}_0(s, t-s) ds = \int_0^t z_1(s) \hat{f}_0(0, t-s) ds,$$

which is bounded by

$$(3.2) \quad \begin{aligned} & C\|f_0\|_{\lambda,p} \int_0^t e^{-\lambda\langle 0,t-s \rangle} \langle t-s \rangle^{-p} |z_1(s)| ds \leq \\ & \leq C\|f_0\|_{\lambda,p} \int_0^t r_{\lambda,p}(s) e^{-\lambda s} \langle s \rangle^{-p} e^{-\lambda(t-s)} \langle t-s \rangle^{-p} ds, \end{aligned}$$

where we used that  $\langle 0, t-s \rangle \geq (t-s)$ . Multiplying by  $e^{\lambda t} \langle t \rangle^p$ , we have the estimate

$$\int_0^t r_{\lambda,p}(s) \frac{\langle t \rangle^p}{\langle t-s \rangle^p \langle s \rangle^p} ds \leq \int_0^t r_{\lambda,p}(s) \left( \frac{1}{\langle s \rangle^p} + \frac{1}{\langle t-s \rangle^p} \right) ds,$$

because  $\langle t \rangle^p \leq C(\langle s \rangle^p + \langle t-s \rangle^p)$ .

The term with  $m = -1$  is bounded by

$$(3.3) \quad \int_0^t |z_{\pm 1}(s)| |h_2(s, t+s)| ds \leq \int_0^t r_{\lambda,p}(s) \|h(s)\|_{\lambda,p} e^{-\lambda s - \lambda\langle t+s \rangle} \langle s \rangle^{-p} \langle t+s \rangle^{-p} ds.$$

We conclude the estimate (3.1) by multiplying by  $e^{\lambda t} \langle t \rangle^p$  and noting that  $\langle t \rangle^p \leq \langle t+s \rangle^p$ .  $\square$

In order to estimate  $\|h\|_{\lambda,p}$ , we need to control its time derivative.

**Proposition 3.2.** *Given  $z_{\pm 1}(t)$ , for  $\lambda, p \geq 0$ ,  $\mathbf{L}_t$  is a continuous operator from  $\mathcal{X}_{\lambda,p+1}$  to  $\mathcal{X}_{\lambda,p}$ :*

$$(3.4) \quad \|\mathbf{L}_t h(t)\|_{\lambda,p} \leq C(r_{\lambda,0}(t) \|h(t)\|_{\lambda,p+1} + r_{\lambda,p}(t) \|h(t)\|_{\lambda,1}).$$

$\mathbf{L}_t$  is also continuous from  $\mathcal{X}_{\lambda',p}$  to  $\mathcal{X}_{\lambda,p}$ , when  $\lambda' > \lambda$ , in fact

$$(3.5) \quad \|f\|_{\lambda,p+1} \leq \frac{1}{\lambda' - \lambda} \|f\|_{\lambda',p}.$$

*Proof.* Recalling the definition of  $A^{\lambda,p}$  in (2.8), we write

$$A_k^{\lambda,p}(\eta)|\mathbf{L}_t h(k, \eta)| \leq \frac{1}{2}|z_1(t)||k| \sum_{m=\pm 1} e^{\lambda\langle k, \eta \rangle} \langle k, \eta \rangle^p |\hat{h}_{k-m}(t, \eta - mt)|;$$

then, by the triangular inequality,

$$\langle k, \eta \rangle \leq \langle k - m, \eta - mt \rangle + \langle m, mt \rangle,$$

and that, when  $|m| = 1$ ,  $\langle m, mt \rangle \leq C + t$ , we have

$$e^{\lambda\langle k, \eta \rangle} \leq C e^{\lambda t} e^{\lambda\langle k-m, \eta-mt \rangle}.$$

Since  $\langle m, mt \rangle$  also verifies  $\langle m, mt \rangle \leq Ct$ , it is true that

$$\langle k, \eta \rangle^p \leq C (\langle k - m, \eta - mt \rangle^p + \langle t \rangle^p),$$

which implies

$$\begin{aligned} A_k^{\lambda,p}(\eta)|\mathbf{L}_t h(k, \eta)| &\leq \\ &\leq C e^{\lambda t} R(t) \sum_{m=\pm 1} e^{\lambda\langle k-m, \eta-mt \rangle} |k| \langle k - m, \eta - mt \rangle^p |\hat{h}_{k-m}(\eta - mt)| + \\ &\quad + C e^{\lambda t} R(t) \sum_{m=\pm 1} e^{\lambda\langle k-m, \eta-mt \rangle} |k| \langle t \rangle^p |\hat{h}_{k-m}(\eta - mt)|. \end{aligned}$$

Using that  $|k| \leq \langle k - m, \eta - mt \rangle$ , we obtain the thesis estimating the first term with

$$Cr_{\lambda,0}(t) \|h(t)\|_{\lambda,p+1}$$

and the second with

$$Cr_{\lambda,p}(t) \|h(t)\|_{\lambda,1}.$$

□

Let us discuss how to choose the norms that will allow us to obtain closed estimates for  $h$  and  $z_{\pm 1}$ . In the Landau Damping type results in the case of Sobolev regularity of order  $\gamma$ , the choice of a suitable Hilbert space  $\mathcal{H}_\gamma$ , with norm  $\|h\|_{\mathcal{H}_\gamma}$ , guarantees that  $L_t$  is a continuous map from  $\mathcal{H}_\gamma$  in the same  $\mathcal{H}_\gamma$  (see [10]). Then the results are achieved estimating, globally in time, a term of the type  $\|h(t)\|_{\mathcal{H}_\gamma}/\langle t \rangle$ , for suitable values of  $\gamma$ , and, correspondingly,  $\langle t \rangle^\gamma R(t)$ . In the case of analytical regularity we can not obtain this behavior and we have to take into account that in (3.4) we can only estimate  $\mathbf{L}_t h$  in a norm that is weaker than the one of  $h$ . We give closed estimates by mixing the typical norms used in Landau-Damping type results with the norms needed for the proof of the abstract Cauchy-Kowalewskaya theorem, following in particular [4].

Given  $\lambda_0 > 0$  and  $a$  such that  $0 < a < 2\lambda_0/\pi$ , for  $t \geq 0$  and  $\lambda < \lambda_0$ , we define the weight

$$(3.6) \quad \beta(t, \lambda) = \beta_a(t, \lambda) = \lambda_0 - \lambda - a \int_0^t \frac{ds}{\langle s \rangle^2} = \lambda_0 - \lambda - a \arctan t.$$

This function is positive for decreasing in time values of  $\lambda$ , and, as in abstract Cauchy-Kowalewskaya theorems, we use it to taking into account the loss of analytical regularity due to the spatial derivative in  $\vartheta$  in the operator  $\mathbf{L}_t$ . In [4] and in other proofs of the abstract Cauchy-Kowalewskaya Theorem, the time dependence of the weight is linear and the solutions exists only for finite time. Here the

Landau-Damping type estimates allow us to choose a weight convergent in time, which, if  $a < 2\lambda_0/\pi$  give the analyticity also for  $t \rightarrow +\infty$ .

More precisely, we define the Banach space  $\tilde{\mathcal{B}}_{a,p}$  as the space of the functions  $h(t)$  such that, if  $\beta(t, \lambda) > 0$ ,  $h(t) \in \mathcal{X}_{\lambda,p}$ . The norm in  $\tilde{\mathcal{B}}_{a,p}$  is

$$(3.7) \quad \|h\|_{a,p} = \sup_{\lambda, t: \beta(t, \lambda) > 0} \beta^{1/2}(t, \lambda) \|h\|_{\lambda,p}.$$

Finally, fixing  $\gamma \geq 3$ , we define the norm

$$(3.8) \quad \|h\|_a = \|h\|_{a,1} + \|h(\cdot)/\langle \cdot \rangle\|_{a,\gamma},$$

and the corresponding Banach space  $\mathcal{B}_a$ , of the function  $h$  with  $\|h\|_a$  bounded. With little abuse of notation, we write:

$$(3.9) \quad \|R\|_a = \sup_{\lambda, t: \beta(t, \lambda) > 0} r_{\lambda,\gamma}(t) = \sup_{\lambda, t: \beta(t, \lambda) > 0} R(t) e^{\lambda t} \langle t \rangle^\gamma$$

Now, we prove the a-priori estimates in  $\mathcal{B}_a$  which allow us to construct the solutions.

**Proposition 3.3.** *Given  $z_{\pm 1}(t)$  with  $\|R\|_a < +\infty$ , if  $h = h(t, \vartheta, \eta)$  solves eq. (2.3) then it satisfies*

$$(3.10) \quad \|h\|_a \leq C \|f_0\|_{\lambda_0, \gamma} + C\mu \|R\|_a \|h\|_a.$$

*Proof.* First we estimate  $\|h(t)\|_{\lambda,1}$ , for  $\lambda$  such that  $\beta(t, \lambda) > 0$ . Using (2.5) and the estimate (3.4) with  $p = 1$ , we have

$$(3.11) \quad \|h(t)\|_{\lambda,1} \leq C \|f_0\|_{\lambda_0} + C\mu \int_0^t r_{\lambda,\gamma}(s) \left( \frac{1}{\langle s \rangle^{\gamma-1}} \frac{\|h(s)\|_{\lambda,\gamma}}{\langle s \rangle} + \frac{1}{\langle s \rangle^{\gamma-1}} \|h(s)\|_{\lambda,1} \right) ds.$$

Multiplying by  $\beta^{1/2}(t, \lambda)$ :

$$(3.12) \quad \beta^{1/2}(t, \lambda) \|h(t)\|_{\lambda,1} \leq C \|f_0\|_{\lambda_0} + C\mu \|h\|_a \|R\|_a \int_0^t \frac{1}{\langle s \rangle^2} \frac{\beta^{1/2}(t, \lambda)}{\beta^{1/2}(s, \lambda)} ds,$$

where we have used that for  $\gamma \geq 3$ ,  $\langle s \rangle^{\gamma-1} \geq \langle s \rangle^2$ . Using that  $\beta(t) \leq \beta(s)$  we estimate the time integral with a constant, then

$$\|h\|_{a,1} \leq C \|f_0\|_{\lambda_0, \gamma} + C\mu \|R\|_a \|h\|_a.$$

Now we estimate  $\|h\|_{\lambda,\gamma}$ : using eq. (2.5) and the estimates (3.4), (3.5) with  $p = \gamma$

$$(3.13) \quad \|h(t)\|_{\lambda,\gamma} \leq C \|f_0\|_{\lambda_0, \gamma} + C\mu \int_0^t r_{\lambda,\gamma}(s) \left( \frac{1}{\langle s \rangle^\gamma} \frac{\|h(s)\|_{\lambda'(s), \gamma}}{\lambda'(s) - \lambda} + \|h(s)\|_{\lambda,1} \right) ds,$$

for any  $\lambda'(s) > \lambda$  such that  $\lambda_0 - \lambda'(s) - a \arctan(s) > 0$ . Dividing by  $\langle t \rangle$  and multiplying by  $\beta^{1/2}(t, \lambda)$ , we obtain

$$(3.14) \quad \frac{\beta^{1/2}(t, \lambda)}{\langle t \rangle} \|h\|_{\lambda,\gamma} \leq C \|f_0\|_{\lambda_0, \gamma} + C\mu \|h\|_a \|R\|_a (I_1 + I_2),$$

where

$$I_1 = \frac{\beta^{1/2}(t, \lambda)}{\langle t \rangle} \int_0^t \frac{ds}{\langle s \rangle^2 \beta^{1/2}(s, \lambda) (\lambda'(s) - \lambda)},$$

$$I_2 = \frac{1}{\langle t \rangle} \int_0^t \frac{\beta^{1/2}(t, \lambda)}{\beta^{1/2}(s, \lambda)} ds.$$

$I_2$  is less than a constant because  $\beta(t, \lambda) \leq \beta(s, \lambda)$ , for  $s \leq t$ .  
In  $I_1$ , we chose  $\lambda' = \lambda'(s) > \lambda$  as

$$\lambda'(s) = \frac{1}{2}(\lambda_0 - a \arctan s) + \frac{\lambda}{2},$$

which verifies

$$\beta(s, \lambda'(s)) = \frac{1}{2}\beta(s, \lambda) > 0,$$

and

$$\lambda'(s) - \lambda = \frac{1}{2}\beta(s) \geq \frac{1}{2}\beta(t) > 0.$$

Then  $I_1$  is bounded by

$$I_1 \leq 2 \frac{\beta^{1/2}(t, \lambda)}{\langle t \rangle} \int_0^t \frac{ds}{\langle s \rangle^2 \beta(s, \lambda)^{3/2}}.$$

Since  $d\beta/ds = -a/\langle s \rangle^2$ , the time integral can be explicitly computed and gives:

$$\int_0^t \frac{ds}{\langle s \rangle^2 \beta(s)^{3/2}} = \frac{2}{a} \left( \frac{1}{\beta^{1/2}(t, \lambda)} - \frac{1}{\beta^{1/2}(0, \lambda)} \right),$$

then also  $I_1$  is less than a constant.  $\square$

Now we estimate  $\|R\|_a$ .

**Proposition 3.4.** *Fixed  $h$  such that  $\|h\|_a < +\infty$ , if  $z_{\pm 1}(t) = R(t)e^{\mp i\varphi(t)}$  solves (2.10), then*

$$(3.15) \quad \|R\|_a \leq C\|f_0\|_{\lambda_0, \gamma}(1 + \mu\|R\|_a) + C\mu\|R\|_a\|h\|_a.$$

*Proof.* We use (3.1) with  $p = \gamma$ : the estimate of the first two terms are obvious; the last one is bounded by  $\mu C\|R\|_a\|h\|_a$  times the integral

$$\int_0^t \frac{ds}{\langle s \rangle^2 \beta^{1/2}(s, \lambda)} = \frac{2}{a} \left( \beta^{1/2}(0, \lambda) - \beta^{1/2}(t, \lambda) \right) \leq \frac{2}{a} \lambda_0^{1/2}$$

$\square$

#### 4. THE MAIN THEOREM

**Theorem 4.1.** *For  $\lambda_0 > 0$  and  $\gamma \geq 3$ , if  $\|f_0\|_{\lambda_0, \gamma}$  is bounded, for  $\mu$  sufficiently small, the unique solution  $h(t, \vartheta, \omega)$  of (2.2) with initial datum  $f_0(\vartheta, \omega)$  verifies  $\|h\|_a < C$  and  $\|R\|_a < C$ .*

*As a consequence,  $R(t) \rightarrow 0$  exponentially fast and there exists  $h_\infty(\vartheta, \omega)$  with  $\|h_\infty\|_{\bar{\lambda}, \gamma} < +\infty$  for some  $\bar{\lambda} > 0$ , such that*

$$f(t, \vartheta + \omega t, \omega) = h(t, \vartheta, \omega) \rightarrow h_\infty(\vartheta, \omega)$$

*exponentially fast.*

*Proof.* We construct the solution with an iterative procedure. For  $n \geq 0$

$$(4.1) \quad h^0(t, \vartheta, \omega) = f_0(\vartheta, \omega)$$

$$(4.2) \quad z_1^n(t) = \hat{h}_1^n(0, t) + \mu \sum_{m=\pm 1} \frac{m}{2} \int_0^t z_m^n(s) \hat{h}_{1-m}^n(s, t - ms) ds,$$

$$(4.3) \quad \widehat{\mathbf{L}_t^n f}(k, \eta) \doteq k \sum_{m=\pm 1} \frac{m}{2} z_m^n(t) \hat{f}_{k-m}(\eta - mt),$$

$$(4.4) \quad \partial_t \hat{h}_k^{n+1}(t, \eta) = \mu \widehat{\mathbf{L}_t^n h^{n+1}}(k, \eta),$$

where in eq. (4.2)  $z_{\pm 1}^n$  is the conjugate of  $z_{\pm 1}^n$ . The linear problems in eq. (4.2) and in eq. (4.4) are easily solvable, and the solutions verify the analogous of the a-priori estimate provided by Prop. 3.3 and Prop. 3.4:

$$\|R^n\|_a \leq C \|f_0\|_{\lambda_0, \gamma} (1 + \mu \|R^n\|_a) + C \mu \|R^n\|_a \|h^n\|_a$$

and

$$\|h^{n+1}\|_a \leq C \|f_0\|_{\lambda_0, \gamma} + C \mu \|R^n\|_a \|h^{n+1}\|_a.$$

Using that  $\|h^0\|_a \leq C \|f_0\|_{\lambda_0, \gamma}$  we can inductively prove that, if  $\mu \|f_0\|_{\lambda_0, \gamma}$  is sufficiently small, then

$$\|h^n\|_a \leq C \|f_0\|_{\lambda_0, \gamma}, \quad \text{and} \quad \|R^n\|_a \leq C \|f_0\|_{\lambda_0, \gamma},$$

uniformly in  $n$ . Choosing  $a' > a$  with  $a' < 2\lambda_0/\pi$ , and estimating the operator  $\mathbf{L}_t^n$  as in Prop. (3.2), we have that all the first derivative of  $h^n$  are uniformly bounded in the region defined by  $\beta_{a'}(t, \lambda) > 0$ ; then, for subsequences,  $h^n$  converges to some  $h \in \tilde{\mathcal{B}}_{a', \gamma}$ . Correspondingly,  $z_{\pm 1}^n$  converges to  $z_{\pm 1}$  with  $\|z_{\pm 1}\|_a$  bounded. The function  $h$  and  $z_{\pm 1}$  solve the coupled equations (2.3) and (2.10). Moreover, putting  $k = 1$  and  $\eta = t$  in (2.5)

$$\hat{h}_1(t, t) = \hat{h}_1(0, t) + \mu \sum_{m=\pm 1} \int_0^t \frac{m}{2} z_m(s) \hat{h}_{k-m}(s, \eta - ms) ds = z_1(t)$$

as follows from (2.10). Then  $h$  solves the non linear equation (2.3), and its uniqueness is guaranteed by the uniqueness of regular solutions (see [15]) (note that the uniqueness implies the convergence to  $h$  and  $z$  for the full sequences  $h_n$  and  $z_n$ ).

Finally, let  $\bar{\lambda} > 0$ , with  $\bar{\lambda} < \lambda_0 - a'\pi/2$ . Then

$$\|\mathbf{L}_t h\|_{0, \gamma} \leq \frac{C}{\bar{\lambda}} \|R\|_{a'} \|h\|_{a'} e^{-\bar{\lambda} t}$$

This inequality implies the existence of

$$\lim_{t \rightarrow +\infty} h(t) = h_\infty,$$

with  $h_\infty \in \mathcal{X}_{\bar{\lambda}, \gamma}$  because  $h \in \mathcal{B}_{a', \gamma}$ . Being  $\gamma \geq 3$ , the norm  $\|h\|_{0, \gamma}$  dominates the sup norm in  $\vartheta$  and  $\omega$ , then  $h(t)$  converges exponentially fast to  $h_\infty$  in the sup norm.

*Remark.* In the analysis carried out in this work the term  $\hat{h}_0(t, \eta) = \frac{\hat{g}(\eta)}{\sqrt{2\pi}}$  can be separated from the other Fourier modes: in the Prop. 3.1 and its following, we can bound separately the zero and nonzero modes, so that, being more careful in the estimates, it is true that

$$\|R\|_a \leq C \left\| f_0 - \frac{g}{2\pi} \right\|_{\lambda_0, \gamma} + C \mu \|g\|_a \|R\|_a + C \mu \|R\|_a \left\| h - \frac{g}{2\pi} \right\|_a.$$



$$\left\| h - \frac{g}{2\pi} \right\|_a \leq C \left\| f_0 - \frac{g}{2\pi} \right\|_{\lambda_0, \gamma} + C\mu \|R\|_a \left\| h - \frac{g}{2\pi} \right\|_a.$$

Using these estimates, to prove our main theorem we need that  $\mu$  is small only w.r.t.  $\|g\|_a$ , so we can keep the interaction fixed and take small only the deviation from the constant phase density. In this way we obtain, with a simple proof, a result similar to [10], in which the authors prove that, for a sufficiently small perturbation of the constant phase density, the solution of the equation (1.1) is asymptotically a free flow and its  $R$  vanishes. The threshold for  $\mu$  that we obtain here could be non optimal, in fact we do not take advantage from a detailed linear analysis as in [9] and [10]. □

## REFERENCES

- [1] Bedrossian J. and Masmoudi N.. Inviscid damping and the asymptotic stability of planar shear flows in the 2d euler equations, 2013.
- [2] Benedetto D. and Caglioti E. and Montemagno U. On the complete phase synchronization for the Kuramoto model in the mean-field limit. *arXiv:1407.6551*, 2014. (To appear in *Comm. Math. Sci.*)
- [3] Benedetto D. and Caglioti E. and Montemagno U. Dephasing of the solutions of the kinetic Kuramoto model towards a fixed asymptotically free state. *arXiv:1411.6304*, 2014. (To appear in *Rend. Mat. and Appl.*)
- [4] R. E. Caflish A simplified version of the abstract Cauchy-Kowalewski Theorem with weak singularities *Bull. Am. Math. Soc.* 23(2):495–500, 1990.
- [5] Caglioti E. and Maffei C. Time asymptotics for solutions of Vlasov-Poisson equation in a circle. *J. Statist. Phys.*, 92(1-2):301–323, 1998.
- [6] Carrillo J.A., Choi Y.P., Ha S.Y., Kang M.J., and Kim Y. Contractivity of transport distances for the kinetic kuramoto equation. *Journal of Statistical Physics*, 156(2):395–415, 2014.
- [7] Choi Y.P., Ha S.Y., Jung S., and Kim Y. Asymptotic formation and orbital stability of phase-locked states for the Kuramoto model. *Phys. D*, 241(7):735–754, 2012.
- [8] Dong J.G. and Xue X.. Synchronization analysis of Kuramoto oscillators. *Commun. Math. Sci.*, 11(2):465–480, 2013.
- [9] Faou E. and Rousset F. Landau damping in Sobolev spaces for the Vlasov-HMF model. *arXiv:1403.1668*, 2014.
- [10] Fernandez B., Varet D.G., and Giacomini G. Landau damping in the kuramoto model, 2014.
- [11] Ha S.Y., Ha T., and Kim J.H. On the complete synchronization of the Kuramoto phase model. *Phys. D*, 239(17):1692–1700, 2010.
- [12] Bedrossian J., Masmoudi N., and Mouhot C. Landau damping: paraproducts and gevrey regularity. *arXiv:1311.2870*, 2013.
- [13] Acebrón J.A., Bonilla L.L., Pérez Vicente C.J., Ritort F., and Renato Spigler. The Kuramoto model: A simple paradigm for synchronization phenomena. *Rev. Mod. Phys.*, 77(137), 2005.
- [14] Kuramoto Y.. Self-entrainment of a population of coupled non-linear oscillators. In Huzihiro Araki, editor, *International Symposium on Mathematical Problems in Theoretical Physics*, volume 39 of *Lecture Notes in Physics*, pages 420–422. Springer Berlin Heidelberg, 1975.
- [15] Lancellotti C.. On the Vlasov limit for systems of nonlinearly coupled oscillators without noise. *Transport Theory Statist. Phys.*, 34(7):523–535, 2005.
- [16] Mirollo R.E. The asymptotic behavior of the order parameter for the infinite-N Kuramoto model. *Transport Theory Statist. Phys.*, 34(7):523–535, 2005.
- [17] Mouhot C. and Villani C. On landau damping. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 2012.
- [18] Strogatz S.H. From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators. *Phys. D*, 143(1-4):1–20, 2000. Bifurcations, patterns and symmetry.
- [19] Strogatz S.H., Mirollo R.E., and Matthews P.C. Coupled nonlinear oscillators below the synchronization threshold: Relaxation by generalized landau damping. *Phys. Rev. Lett.*, 68:2730–2733, May 1992.

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